The matrix of the operator. Calculation of their Eigenvalues and eigenvectors. Bringing a quadratic form to the canonical form

P.S.: Linear operators Example 2

- In the basis e1, e2, the operator (transformation) $\tilde{A}$ has the matrix $A = \begin{vmatrix} 17 & 6 \\ 6 & 8 \end{vmatrix}$. Find the matrix of the operator $\tilde{A}$ in the basis $e_1^* = e_1 - 2e_2 \; and \; e_2^* = 2e_1 + e_2$.

Solution

- $C = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} \Rightarrow C^{-1} = \frac{1}{5}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$

$$A^* = C^{-1}AC$$

- $A^* = \frac{1}{5}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}\begin{vmatrix} 17 & 6 \\ 6 & 8 \end{vmatrix}\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} \Rightarrow \begin{vmatrix} 5 & 0 \\ 0 & 20 \end{vmatrix}$

Eigenvectors and eigenvalues of a linear operator

- Definition. A vector $x \neq 0$ is called an eigenvector of a linear operator $\tilde{A}$ if there is a number λ such that

$$\tilde{A}(\boldsymbol{x}) = \lambda \boldsymbol{x}.$$

- The number λ is called the eigenvalue of the operator $\tilde{A}$ (matrix A) corresponding to the vector x.

$$|A - \lambda E| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{2n} \\ a_{n1} & a_{n2} & a_{nn} - \lambda \end{vmatrix} = 0.$$

Example

- Find the eigenvalues and eigenvectors of the linear operator $\tilde{A}$ given by the matrix

$$A = \begin{pmatrix} 1 & 4 \\ 9 & 1 \end{pmatrix}.$$

Solution

- $|A - \lambda E| = \begin{vmatrix} 1-\lambda & 4 \\ 9 & 1-\lambda \end{vmatrix} = 0$
- $\lambda^2 - 2\lambda - 35 = 0$ =>
- => $D = (-2)^2 - 4 * 1 * (-35) = 144$ => $x_{1,2} = \frac{-(-2)\pm\sqrt{144}}{2*1} = -5\ and\ 7$

$$(A-\lambda_1 E)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0} \quad \text{или} \quad \begin{pmatrix} 6 & 4 \\ 9 & 6 \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

- => $\begin{cases} 6x_1 + 4x_2 = 0 \\ 9x_1 + 6x_2 = 0 \end{cases} \Rightarrow -3x_1 = 2x_2 \Rightarrow x_2 = -1{,}5x_1$
- Putting $x_1 = c$, we get that the vectors $x^{(1)} = (c; -1{,}5c)$ for any $c \neq 0$ are eigenvectors of the linear operator $\tilde{A}$ with eigenvalue $\lambda = -5$.

Solution

- Similarly, one can verify that the vectors $x^{(2)} = \left(\frac{2}{3}c_1; c_1\right)$ for any $c_1 \neq 0$ are eigenvectors of the linear operator $\tilde{A}$ with eigenvalue $\lambda_2 = 7$.

Example

- Bring the matrix $A=\begin{pmatrix}1 & 4\\ 9 & 1\end{pmatrix}$. of the linear operator $\tilde{A}$ to a diagonal form.

Solution

- In previous example we found that $\lambda_1 = -5\ and\ \lambda_2 = \ 7$
- $x^{(1)} = (c; -1{,}5c)$ and $x^{(2)} = \left(\frac{2}{3}c_1; c_1\right)$ for any c $\neq$ 0 and $c_1 \neq 0$ (c=2, $c_1 = 6$)
- $x^{(1)} = (2; -3) and\ x^{(2)} = (4; 6)$

- Matrix $A^* = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} \Rightarrow A^* = \begin{vmatrix} -5 & 0 \\ 0 & 7 \end{vmatrix}$

Solution

- $C = x^{(n)} \Rightarrow \begin{vmatrix} 2 & 4 \\ -3 & 6 \end{vmatrix} \Rightarrow \quad A^{*} = C^{-1}AC$
- $A^{*} = \begin{vmatrix} 2 & 4 \\ -3 & 6 \end{vmatrix}^{-1} \begin{vmatrix} 1 & 4 \\ 9 & 1 \end{vmatrix} \begin{vmatrix} 2 & 4 \\ -3 & 6 \end{vmatrix} \Rightarrow \frac{1}{24} \begin{vmatrix} 6 & -4 \\ 3 & 2 \end{vmatrix} \begin{vmatrix} -10 & 28 \\ 15 & 42 \end{vmatrix} =$
- $\frac{1}{24} \begin{vmatrix} -60-60 & 168-168 \\ -30+30 & 84+84 \end{vmatrix} = \frac{1}{24} \begin{vmatrix} -120 & 0 \\ 0 & 168 \end{vmatrix} = \begin{vmatrix} -5 & 0 \\ 0 & 7 \end{vmatrix}$

Quadratic forms

- Definition. The quadratic form of $L = (x_1, x_2, \dots x_n)$ in n variables is the sum, each term of which is either the square of one of the variables, or the product of two different variables, taken with some coefficient:

$$L(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

$$L = X'AX,$$

$$A^* = C'AC.$$

Example

- Given a quadratic form $L = (x_1, x_2, x_3) = 4x_1^2 - 12x_1x_2 - 10x_1x_3 + x_2^2 - 3x_3^2$. Write it down in matrix form.

Solution

- $L = (x_1, x_2, x_3)\begin{vmatrix} 4 & -6 & -5 \\ -6 & 1 & 0 \\ -5 & 0 & -3 \end{vmatrix}\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}$

Example

- Given a quadratic form $L = (x_1, x_2\ ) = 2x_1^2 + 4x_1x_2 - 3x_2^2$. Find a quadratic form $L = (y_1, y_2\ )$ obtained from the given by a linear transformation $x_1 = 2y_1 - 3y_2\ and\ x_2 = y_1 + y_2$.

Solution

- $A = \begin{vmatrix} 2 & 2 \\ 2 & -3 \end{vmatrix}$
- $C = \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix}$

$$A^* = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 13 & -17 \\ -17 & 3 \end{pmatrix},$$

- $L(y_1, y_2) = 13y_1^2 - 34y_1y_2 + 3y_2^2$

Example

- Prove that the quadratic form $L = (x_1, x_2\ ) = 13x_1^2 - 6x_1x_2 + 5x_2^2$ is positive definite.

Solution 1

$$A = \begin{pmatrix} 13 & -3 \\ -3 & 5 \end{pmatrix}.$$

$$|A - \lambda E| = \begin{vmatrix} 13-\lambda & -3 \\ -3 & 5-\lambda \end{vmatrix} \text{ или } \lambda^2 - 18\lambda + 56 = 0.$$

$$\lambda_1 = 14, \ \lambda_2 = 4.$$

Since the roots of the characteristic equation of the matrix A are positive, then, on the basis of the above theorem, the quadratic form L is positive definite.

Solution 2

- Since the principal minors of the matrix A are positive, then, according to the Sylvester criterion, this quadratic form L is positive definite.

$$|a_{11}| = 13, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 13 & -3 \\ -3 & 5 \end{vmatrix} = 56$$

Example

- Given vectors $a = \alpha m + \beta n \; and \; b = \gamma m + \delta n, where \; |m| = k; \; |n| = l; (\widehat{m, n}) = \varphi$. Find

$a)$ $(\lambda a + \mu b)(\upsilon a + \tau b)$

$b)$ $\text{пр}_{\text{ь}}(\lambda a + \tau b)$

c) $\cos(\widehat{a, \tau b})$

1.1. $\alpha = -5$, $\beta = -4$, $\gamma = 3$, $\delta = 6$, $k = 3$, $l = 5$, $\varphi = 5\pi/3$, $\lambda = -2$, $\mu = 1/3$, $\nu = 1$, $\tau = 2$. (*Ответ*: а) 2834.)

Solution

- $a = -5m - 4n$
- $b = 3m + 6n$
- $|m| = 3;$
- $|n| = 5;$
- $(\widehat{m, n}) = \varphi \Rightarrow |\bar{m}||\bar{n}|cos\frac{5\pi}{3} = 3 * 5 * \left(\frac{1}{2}\right) = \frac{15}{2}.$

Solution a

- $(\lambda a + \mu b)(\upsilon a + \tau b) \Rightarrow \left(-2a + \frac{1}{3}b\right)(a + 2b) = -2a^2 - 4ab + \frac{1}{3}ab + \frac{2}{3}b^2 = -2a^2 - \frac{11}{3}ab + \frac{2}{3}b^2$

- $-2(-5m - 4n)^2 - \frac{11}{3}(-5m - 4n)(3m + 6n) + \frac{2}{3}(3m + 6n)^2 = -2(25m^2 + 40mn + 16n^2) - \frac{11}{3}(-15m^2 - 42mn - 24n^2) + \frac{2}{3}(9m^2 + 36mn + 36n^2) \Rightarrow -50m^2 - 80mn - 32n^2 + 55m^2 + 154mn + 88n^2 + 6m^2 + 24mn + 24n^2 = 11m^2 + 98mn + 80n^2 = 11 * (3)^2 + 98 * \frac{15}{2} + 80 * (5)^2 = 99 + 735 + 2000 = 2834$

Solution b

- $\text{пр}_\text{ь}(\lambda a + \tau b) \Rightarrow c = (-2a + 2b) \Rightarrow -2(-5m - 4n) + 2(3m + 6n) \Rightarrow 10m + 8n + 6m + 12n \Rightarrow 16m + 20n$

$$\text{пр}_\mathbf{b}\mathbf{c} = \frac{\mathbf{c} \cdot \mathbf{b}}{|\mathbf{b}|},$$

- $c * b = (16m + 20n)(3m + 6n) = 48m^2 + 156mn + 120n^2 \Rightarrow 432 + 1170 + 3000 = 4602$
- $|b| = \sqrt{b^2} = \sqrt{(3m + 6n)^2} = \sqrt{9m^2 + 36mn + 36n^2} \Rightarrow \sqrt{81 + 270 + 900} = \sqrt{1251}$
- $\text{пр}_\text{ь}(-2a + 2b) = 4602/\sqrt{1251}$

Solution c

- $\cos(\widehat{a, \tau b}) \Rightarrow \cos((-5m - 4n), 2(3m + 6n))$

$$\cos(\widehat{\mathbf{d}, \mathbf{e}}) = \frac{\mathbf{d} \cdot \mathbf{e}}{|\mathbf{d}||\mathbf{e}|},$$

- $d = (-5m - 4n)\ and\ e = \ 2(3m + 6n)$
- $d * e = -30m^2 - 54mn - 24n^2 \Rightarrow -270 - 405 - 600 = -1275$
- $|d| = \sqrt{(-5m - 4n)^2} = \sqrt{25m^2 + 40mn + 16n^2} \Rightarrow \sqrt{225 + 300 + 400} = \sqrt{925}$
- $|e| = \sqrt{(6m + 12n)^2} = \sqrt{36m^2 + 144mn + 144n^2} = \sqrt{324 + 1080 + 3600} = \sqrt{5004}$
- $\cos((-5m - 4n), 2(3m + 6n)) = \frac{-1275}{\sqrt{925} * \sqrt{5004}} \approx -0{,}58$